

Recovery of Dirac Equations From Their Solutions*

Kurt Just[‡] Zbigniew Oziewicz[†] and Erwin Sucipto[§]

Received November 22, 1999

We deal with quantum field theory in the restriction to external Bose fields. Let $(i\gamma^\mu \partial_\mu - \mathcal{B})\psi = 0$ be the Dirac equation. We prove that a non-quantized Bose field \mathcal{B} is a functional of the Dirac field ψ , whenever this ψ is strictly canonical. Performing the trivial verification for the $\mathcal{B} := m = \text{constant}$ which yields the free Dirac field, we also prepare the tedious verifications for all \mathcal{B} which are non-quantized and static. Such verifications must not be confused, however, with the easy and rigorous proof of our formula, which is shown in detail.

Key words: quantum field theory; Clifford algebra, canonical Dirac field, the Dirac operator, Bose field
1995 Physics and Astronomy Classification Scheme: 03.70.+k
2000 Mathematics Subject Classification: 81E10

1. DIFFERENT DIRAC THEORIES

A canonical Dirac field ψ is never an operator, but an operator valued distribution (Jost 1965). For such a four component spinor, its hermitian conjugate ψ^\dagger and its transpose ψ^T , we postulate the anti-commutators

$$[\psi(x), \psi(y)^\dagger]_+ \delta(x^0 - y^0) = \delta(x - y), \quad [\psi(x), \psi(y)^T]_+ \delta(x^0 - y^0) = 0, \quad (1)$$

where $\delta(z) := \delta(z^0)\delta(z^1)\delta(z^2)\delta(z^3)$. We consider the Dirac equation

$$\{i\partial^x - \mathcal{B}(x)\}\psi(x) = 0 \quad \text{with} \quad \partial^x := \gamma^\mu \partial / \partial x^\mu \quad \text{and} \quad \mathcal{B} = \gamma_0 \mathcal{B}^\dagger \gamma_0 \quad (2)$$

(the latter makes the action $\int \bar{\psi}(i\partial - \mathcal{B})\psi$ hermitian). Evidently, \mathcal{B} is a member of Dirac's Clifford algebra \mathcal{Cl}_D . Since this aspect is irrelevant here, we need not choose

* This paper is in a final form and no version of it will be submitted for publication elsewhere.

[‡] The University of Arizona, Department of Physics, Tucson AZ 85721, just@physics.arizona.edu

[†] Universidad Nacional Autónoma de México, Facultad de Estudios Superiores, C.P. 54700 Cuautitlán Izcalli, Apartado Postal # 25, Estado de México, oziewicz@servidor.unam.mx and Uniwersytet Wrocławski, Instytut Fizyki Teoretycznej, plac Maksa Borna 9, 50204 Wrocław, Poland. Supported by el Consejo Nacional de Ciencia y Tecnología (CONACYT) de México, Grant # 27670 E (1999-2000) and by UNAM, DGAPA, Programa de Apoyo a Proyectos de Investigación e Innovación Tecnológica, Proyecto IN-109599 (1999-2000). Member of Sistema Nacional de Investigadores de México, Expediente # 15337.

[§] University of Arizona, Department of Physics, Tucson AZ 85721 and Institut Teknologi Bandung, Indonesia, sucipto@physics.arizona.edu.

a specific *representation* of Dirac's γ^μ and not even a basis in $\mathcal{C}\ell_D$. As another member of this, we need the time ordered product

$$b(x, z) := (4\pi)^2 T\psi(x+z)\overline{\psi}(x-z) \in \mathcal{C}\ell_D. \quad (3)$$

The covariant ordering needed here must directly act only on basic fields, not on their derivatives. Hence the time ordering of the latter must be *defined* by

$$T\phi_{,\mu}(x)\chi(y) := \partial_\mu^x T\phi(x)\chi(y).$$

This (widely used, but rarely emphasized) prescription has been explained by (Nambu 1952), (Callan *et al.* 1970), (DeWitt 1984), (Just & The 1986), (Sterman 1993), (DeWitt - Morette 1994). Of the canonical relations (1), only the first will be used here; but both are needed to define ψ completely.

Further treatment of \mathcal{B} and b can proceed in 4 ways, of which only the last one will be pursued here:

- (a) One may *desire* that \mathcal{B} also be a canonical field. This gives the usual 'effective' field theory (Weinberg 1995/96). There one starts from (1) and (2) and their extensions to Bose fields; but all these break down under the infinite renormalization (Brandt 1969). Hence that desire, explained in the introductions of many books on quantum fields, is only satisfied as long as one does not admit interactions.
- (b) All divergencies are prevented in Quantum Induction (QI), where \mathcal{B} is a non-canonical quantum field (Just & Sucipto 1997). For this unconventional theory, peripheral results have been explained briefly, but only at the expense of setting aside the proofs (Just & Thevenot 2000, Just *et al.* 2000).
- (c) Some divergencies are also avoided when one restricts \mathcal{B} to be non-quantized forever. This is done in the mathematical theory of heat kernels (Esposito 1998), where one studies elaborately the boundary conditions for (3) at large separations z .
- (d) In this paper, we examine a simple consequence of the strict postulates (1) and (2). It also holds in (b), but now we prove it only for *non-quantized* \mathcal{B} (for clarity excluded from QI); hence the present proof holds as well for (c). We nevertheless do not apply heat kernels, because 'outer' boundary conditions on (3) are superfluous here.

For the case (d), we prove in Section 2 the explicit recovery of \mathcal{B} from (2) as a functional of ψ . The result is verified for the constant $\mathcal{B} = m$ in Section 3. Restricting the non-quantized \mathcal{B} to a static $\beta(\vec{x})$ in Section 4, we *prepare* its recovery in Section 5.

2. THE RECOVERY FORMULA

In what follows,

$$\not{z}_-^{-3} := (z^2 - i\epsilon)^{-2} \not{z} \quad \text{with} \quad \epsilon \rightarrow +0. \quad (4)$$

For (3), the canonical postulates (1) and (2) imply

$$\{\not{\partial}^x + \not{\partial}^z + 2i\mathcal{B}(x+z)\} b(x, z) = 2\pi^2 \delta(z) = i\not{\partial}^z \not{z}_-^{-3}. \quad (5)$$

Here we have used (4) in order to give to (3) the analyticity of a time ordered product. At this point, it will be useful to introduce

$$r(x, z) := b(x, z) - i \not{x}_-^{-3} . \quad (6)$$

We shall see that this *remainder* is less singular than \not{x}_-^{-3} for $z \rightarrow 0$. Using (6) in (5), we obtain

$$2\mathcal{B}(x+z) \not{x}_-^{-3} = \{ \not{\partial}^x + \not{\partial}^z + 2i\mathcal{B}(x+z) \} r(x, z) \quad (7)$$

$$\approx \not{\partial}^z r(x, z) = \not{\partial}^z [b(x, z) - i \not{x}_-^{-3}] . \quad (8)$$

It is essential that the *equalities* in (7) and (8) hold strictly, whereas the left side of (8) only approximately equals the right side of (7). In (8) we have used that $z \rightarrow 0$ makes the remainder $r(x, z)$ singular, such that the strongest singularity on the right of (7) is contained in $\not{\partial}^z r(x, z)$. Comparing the left sides of (7) and (8), we have seen that $r(x, z)$ is less singular than \not{x}_-^{-3} ; then we eliminated it by (6). The resulting $\not{\partial}^z \not{x}_-^{-3} = \text{const} \cdot \delta(z)$, however, drops out when we multiply (7) and (8) by \not{x}^3 , obtaining

$$2\mathcal{B}(x) + \dots = [\not{\partial}^z b(x, z) + \dots] \not{x}^3 . \quad (9)$$

The dots symbolize terms which we have neglected in (8) or in the approximation $\mathcal{B}(x+z) \approx \mathcal{B}(x)$. All these terms contribute nothing to (9) with $z \rightarrow 0$; hence

$$2\mathcal{B}(x) = \lim_{z \rightarrow 0} [\not{\partial}^z b(x, z)] \not{x}^3 . \quad (10)$$

While (9) is a quantum field, its local limit (10) is non-quantized, because we assumed this in (2). Noting (3), we see that (10) has proved

$$\mathcal{B}(x) = 8\pi^2 \lim_{z \rightarrow 0} [\not{\partial}^z T\psi(x+z) \overline{\psi}(x-z)] \not{x}^3 = \gamma_0 \mathcal{B}(x)^\dagger \gamma_0 . \quad (11)$$

The second assertion follows when we start from (5) with the differential operator replaced by one which acts on the bilocal field $b(x, z)$ from the *right* side.

In (11) the multiplication by $\not{x}^3 \rightarrow 0$ has removed the singularity. Therefore, the step functions in the time ordering need not be differentiated. Hence the $\not{\partial}^z$ can be expressed by operators acting on x , giving

$$\mathcal{B}(x) = 8\pi^2 \gamma^\mu \lim_{z \rightarrow 0} T\psi(x-z) \overleftrightarrow{\partial}_\mu^x \overline{\psi}(x+z) \not{x}^3 . \quad (12)$$

Here we need no longer indicate that no differentiation acts on \not{x}^3 . Thus we have *recovered* the non-quantized Bose field with which Dirac's equation (2) has been solved, provided this has been done by a Dirac field ψ satisfying (1).

In this paper we ask to what extent (11) can be verified by two examples:

1. $\mathcal{B} = m = \text{constant}$, which yields the free ψ .
2. Static non-quantized $\mathcal{B}(x) := \beta(\vec{x})$.

Since neither of these examples is a quantum field, the assumptions of heat kernels are valid here (Esposito 1998). For the free Dirac field, we verify in Section 3 the recovery of $\mathcal{B} = m$ by (11). For non-quantized and static \mathcal{B} , the complete solution ψ of (1) and (2) follows in principle from an eigenvalue problem in *three* dimensions. For such a case, we make the functional (11) more specific in Section 4. The following Sections 3-7 describe both a very easy and an extremely difficult verification of (11). Its rigorous proof (under the conditions of Section 1d) is *completed* at (10).

3. A SIMPLE VERIFICATION

Let us define $\delta(p, q)$ such that the *measure*

$$d(p) := (2\pi)^{-3} \theta(p_0) \delta(p^2 - m^2) dp \quad (13)$$

over the sharp mass shell $p_0 = \sqrt{\vec{p}^2 + m^2}$ makes

$$\int f(p) d(p) \delta(p, q) = f(q) \quad \text{for} \quad q_0 = \sqrt{\vec{q}^2 + m^2} .$$

For $m = \text{const} > 0$, the non-quantized spinors $u_\sigma(p)$ with helicity label σ are to fulfill

$$(\not{p} \mp m) u_\sigma^\pm(p) = 0 \quad \text{and} \quad \sum_\sigma u_\sigma^\pm(p) \overline{u}_\sigma^\pm(p) = \not{p} \pm m .$$

With the Poincaré invariant vacuum $|\rangle$, we postulate

$$a_\sigma^\pm(p) |\rangle = 0 \quad \text{and} \quad [a_\sigma^\pm(p), a_\tau^\pm(q)^\dagger]_+ = \delta_{\sigma\tau} \delta(p, q) .$$

All other anti-commutators of the $a_\sigma^\pm(p)$ are assumed to vanish.

Then (1) and (2), with $\mathcal{B} = m = \text{constant}$, are satisfied by the free canonical Dirac field,

$$\psi(x) = \sum_\sigma \int \left\{ e^{-ipx} u_+^\sigma(p) a_\sigma^+(p) + e^{+ipx} u_-^\sigma(p) a_\sigma^-(p)^\dagger \right\} d(p) . \quad (14)$$

Its familiar propagator will be needed in the form

$$\begin{aligned} (2\pi)^4 \langle |T\psi(2z)\overline{\psi}(0)| \rangle &= i \int e^{-2ipz} dp (\not{p} + i\epsilon - m)^{-1} \quad (\text{with } \epsilon \rightarrow +0) \\ &= \pi^2 (i \not{z}^3 - m z_-^2 + \dots) . \end{aligned} \quad (15)$$

Since (11) is non-quantized, it equals its expectation value in any state such as $|\rangle$. Hence (11) is *verified* by (15), because it yields

$$\mathcal{B}(x) = 8\pi^2 \lim_{z \rightarrow 0} \langle |\not{\partial}^z T\psi(2z)\overline{\psi}(0)| \rangle \not{z}^3 = m . \quad (16)$$

No further solution of Dirac's equation (2) is known for which (11) can be verified as easily.

4. STATIC BACKGROUNDS

When \mathcal{B} is not only non-quantized, but also time independent, we define

$$\beta(\vec{x}) := \mathcal{B}(x) . \quad (17)$$

Let the spinors $u^\sigma(x)$ solve the eigenvalue problem

$$H u^\sigma = \omega_\sigma \cdot u^\sigma \quad \text{with} \quad H := \gamma_0 \{ \beta(\vec{x}) - i \not{\partial}^x \} . \quad (18)$$

Although we use $\not{\partial}^x = \gamma^\mu \partial / \partial x^\mu$ to avoid additional notations, (18) involves only $x^r \in \{x^1, x^2, x^3\}$, because β and u^σ are independent of x^0 . Since (11) for (17) makes $\beta^\dagger \gamma_0 = \gamma_0 \beta$, the operator H is *hermitian*. Hence its eigenvalues ω_σ are real and the solutions can be made orthogonal:

$$\int u^\sigma(\vec{x})^\dagger d^3x u^\tau(\vec{x}) = \delta^{\sigma\tau} \quad \text{for} \quad \sigma, \tau \in \{1, 2, \dots\} . \quad (19)$$

For brevity we use notations suitable for a discrete frequency spectrum, although that of (18) will often be continuous as in (14), or mixed as in the hydrogen atom. In either case, the σ in (18) takes infinitely many values in contrast to (14), where it labels two helicities. In addition, we assume

$$\sum_\sigma u^\sigma(\vec{x}) u^\sigma(\vec{y})^\dagger = \delta^3(\vec{x} - \vec{y}) . \quad (20)$$

This *completeness* relation will be most important here. It is compatible with (19), but not implied by this. Then Dirac's equation (2) is satisfied by each term of

$$\psi(x) = \sum_\sigma e^{-i\omega_\sigma x^0} u^\sigma(\vec{x}) a_\sigma . \quad (21)$$

We find (1) satisfied when we make (21) a quantum field by postulating

$$[a_\sigma, a_\tau^\dagger]_+ = \delta_{\sigma\tau} \quad \text{and} \quad [a_\sigma, a_\tau]_+ = 0 . \quad (22)$$

5. DESIRABLE VERIFICATIONS

We specify a ground state $|\cdot\rangle$ by separating positive and negative frequencies in (21):

$$\psi(x) = \sum_\sigma e^{-i\omega_\sigma x^0} u_+^\sigma(\vec{x}) a_\sigma + \sum_\tau e^{+i\Omega_\tau x^0} u_-^\tau(\vec{x}) b_\tau^\dagger , \quad (23)$$

$$a_\sigma |\cdot\rangle = 0 \quad \text{and} \quad b_\tau |\cdot\rangle = 0 ,$$

where $\omega_\sigma > 0$ and $\Omega_\tau := -\omega_\tau > 0$. Rewriting the anti-commutators (22) in the notation (23), we deduce the *propagator*

$$\begin{aligned} F(x, z) &:= \langle \cdot | T \psi(x+z) \overline{\psi}(x-z) | \cdot \rangle \\ &= \theta(z^0) \sum_\sigma e^{-2i\omega_\sigma z^0} u_+^\sigma(\vec{x}+\vec{z}) \overline{u}_+^\sigma(\vec{x}-\vec{z}) \\ &\quad - \theta(-z^0) \sum_\tau e^{+2i\Omega_\tau z^0} u_-^\tau(\vec{x}+\vec{z}) \overline{u}_-^\tau(\vec{x}-\vec{z}) , \end{aligned} \quad (24)$$

which unlike (15) is not Poincaré covariant. Having restricted the field β in (17) to become non-quantized, we find it equal to its expectation value

$$\beta(\vec{x}) = \langle \cdot | \mathcal{B}(x) | \cdot \rangle = 8\pi^2 \lim_{z \rightarrow 0} [\not{\partial}^z F(x, z)] \not{z}^3 . \quad (25)$$

Since (11) is the same as (12), we can in (25) with (24) omit those terms in which the $u_{\pm}^{\sigma}(\vec{x})$ are not differentiated. Returning from (23) to the compact notation (21), we obtain

$$\beta(\vec{x})/8\pi^2 = \gamma^r \lim_{z \rightarrow 0} \sum_{\sigma} u^{\sigma}(\vec{x} - \vec{z}) \overset{\leftrightarrow}{\partial}_r^x \overline{u}^{\sigma}(\vec{x} + \vec{z}) \{ \theta(-z_0)\theta(\omega_{\sigma}) - \theta(z_0)\theta(-\omega_{\sigma}) \} e^{2i\omega_{\sigma}z_0} \not{z}^3 . \quad (26)$$

In all the limits taken in (10) through (26), $z = 0$ may be approached on any line through Minkowski space which does not touch the cone $z^2 = 0$. Hence (26) can be specialized in many ways. Starting with $\vec{z} \equiv 0$, for instance, we see that the matrices

$$u^{\sigma}(\vec{x}) \overset{\leftrightarrow}{\partial}_r^x \overline{u}^{\sigma}(\vec{x}) e^{2i\omega_{\sigma}z_0}$$

must *increase* so strongly that their sums behave as $(z_0)^{-3}$ for $z_0 \rightarrow \mp 0$ and $\omega_{\sigma} \rightarrow \pm\infty$. Alternatively, we may start with $z_0 \equiv \mp 0$, so that (26) simplifies to

$$\begin{aligned} \beta(\vec{x})/8\pi^2 &= \gamma^r \lim_{z \rightarrow 0} \sum_{\sigma} \theta(\omega_{\sigma}) u^{\sigma}(\vec{x} - \vec{z}) \overset{\leftrightarrow}{\partial}_r^x \overline{u}^{\sigma}(\vec{x} + \vec{z}) (\gamma_s z^s)^3 \\ &= -\gamma^r \lim_{z \rightarrow 0} \sum_{\sigma} \theta(-\omega_{\sigma}) u^{\sigma}(\vec{x} - \vec{z}) \overset{\leftrightarrow}{\partial}_r^x \overline{u}^{\sigma}(\vec{x} + \vec{z}) (\gamma_s z^s)^3 . \end{aligned} \quad (27)$$

Here as in (23) through (26), the sum runs either over all solutions of (18) with frequencies $\omega_{\sigma} > 0$ or over those with $\omega_{\sigma} < 0$.

6. GENERAL REMARKS

In the Coulomb field of a proton, (24) results from all the spinors u^{σ} of either an electron or a positron. For their partly continuous spectra, suitable *notations* must be invented, because we have for brevity used those for discrete ω_{σ} . In either case, however, the result must verify

$$\beta(\vec{x}) = m + \gamma_0 \frac{e}{|\vec{x}|} . \quad (28)$$

Since the non-quantized and static fields (17) include the $\mathcal{B} = m$ of the free Dirac field (14), the $\beta = m$ must also follow from (26). However, verifying this will be more difficult than under the manifest Lorentz covariance employed in Section 3. The greatest obstacle to any use of (26) is that it requires *infinitely* many exact solutions of (18).

Thus we have performed one of those verifications which are possible as indicated in Section 5 (namely that of (28) with $\epsilon = 0$); but we did so in a much simpler way. The verification shown in Section 3 consists of the *single* line (16), because (13 - 15) merely state our notations for widely familiar objects. Having tried to evaluate (26) for (18) with $\beta(\vec{x}) = m$, we know that doing so will cost much work. Hence that attempt has shown that a problem which under Lorentz covariance is trivial can be poorly tractable when this is not manifest.

Let us also remark that all this does not concern a physical theory. It rather forms a didactic *simplification* (by non-quantized Bose fields) of a mathematical result from QI. This new version of Quantum Field Theory has only recently been suggested (Just & Sucipto 1997). Hence the proof of (12) for quantum fields \mathcal{B} must be deferred until publication of QI.

7. RESULTS AND EXPECTATIONS

Whereas (28) provides one of the few simple problems in which all solutions of (18) are known, (26) must hold for every $\beta(\vec{x})$ admitted here. For known as well as unknown u^σ , we thus obtain the

Recovery Theorem: *Whenever the solutions $u^\sigma(\vec{x})$ of Dirac's equation (18) with any non-quantized and time independent matrix $\beta(\vec{x}) \in \mathcal{Cl}_D$ fulfill the completeness relation (20), that field $\beta(\vec{x})$ is recovered by (26).*

Comparing this result with the familiar 'inverse scattering' theory (Bertero & Pike 1992), we see that in some respect the opposite is done there. One wants to derive *approximations* to a potential by using as few as possible of its consequences. On the contrary, we recover $\beta(\vec{x})$ exactly by (26), but only when the exact solutions $u^\sigma(\vec{x})$ of (18) are known (either for all $\omega_\sigma > 0$ or for all $\omega_\sigma < 0$). The further analysis of (5) reveals that (12) must satisfy consistency conditions, such as Dirac induced field equations and the absence of Pauli terms (Just & Thevenot 2000); but these do not invalidate the present results.

In our derivation, we have used quantum field theory (Jost 1965) in the restriction to external Bose fields (Esposito 1998). However, the resulting 'solution' of (18) with the Bose field (17) does not involve quantum fields and not even time coordinates. Thus it should equally well be of interest to readers who treat in Dirac's equation (2) not only the matrix \mathcal{B} but also the spinor ψ as *non-quantized* fields (Thaller 1992). For this case (in which (1) is ignored), our general result (12) might not be needed, if one merely wants to derive (26) from (18 - 20), hence without (21 - 25). Thus there remains the

Question: *Is there a simpler way to prove (26), or will our approach remain the best method to reach that result about classical solutions of the time independent Dirac equation (18)?*

Acknowledgements

For comments we are thankful to A. Borowiec, N. Ercolani, K. Kwong, C. Levermore, W. Stoeger, and J. Thevenot.

References

- Bertero M. and E.R. Pike (1992). *Inverse Problems in Scattering and Imaging*, Hilger, Bristol
- Brandt R. (1969). *Annals of Physics* **52**, pp. 122-175
- Esposito G. (1998). *Dirac Operators and Spectral Geometry*, Cambridge University Press
- Callan C.G., S. Coleman, R. Jackiw (1970). *Annals of Physics* **59**, pp. 42-73, eq.(2.9)
- DeWitt B.S. (1984). *Super-Manifolds*, University Press, Cambridge, p.246
- DeWitt-Morette C. (1994). "Functional Integration", in: Flato, Kerner, Lichnerowicz, editors, *Physics on Manifolds*, Kluwer, Dordrecht, p. 66-91
- Jost R. (1965). "The General Theory of Quantized Fields", American Mathematical Society, Providence Rhode Island
- Just K., K. Kwong and Z. Oziewicz (2000). "Light as Caused Neither by Bound States nor by Neutrinos", in: Valeri V. Dvoeglazov, editor, *Lorentz Group, CPT, and Neutrinos*, World Scientific Publishing, Singapore, submitted
- Just K. and E. Sucipto (1997). "Basic versus Practical Quantum Induction", in: Doeblner, Scherer and Schulte, editors, *Group 21*, World Scientific, Singapore, vol. 2, p.739
- Just K. and L. S. The (1986). *Foundations of Physics* **16**, pp. 1127-1140, Appendix B
- Just K. and J. Thevenot (2000). "Pauli Terms Must be Absent in Dirac's Equation", in: Rafał Ablamowicz, editor, *Clifford Algebras and Their Applications in Mathematical Physics*, Birkhäuser, Boston, submitted
- Nambu Y. (1952). *Progress in Theoretical Physics* **7**, pp. 121-170, eq.(3.9)
- Sterman, G. (1993). *An Introduction to Quantum Field Theory*, University Press, Cambridge, p. 114
- Thaller B. (1992). *The Dirac Equation*, Springer, Berlin
- Weinberg S. (1995/96). *The Quantum Theory of Fields*, Cambridge University Press